In a nutshell: The condition number

Given a system of linear equations $A\mathbf{u} = \mathbf{v}_0$ that has a unique solution $\mathbf{u}_0$, then suppose there is a small error in the right-hand side, so we are actually solving $A\mathbf{u} = \mathbf{v}_0 + \Delta\mathbf{v}$. We will answer how close the solution to this modified system of linear equations is to the solution $\mathbf{u}_0$.

1. Assume that the solution we find is $\mathbf{u}_0 + \Delta\mathbf{u}$.
2. Then $\frac{\|\Delta\mathbf{u}\|_2}{\|\mathbf{u}_0\|_2} \le \operatorname{cond}(A)\frac{\|\Delta\mathbf{v}\|_2}{\|\mathbf{v}_0\|_2}$; that is, the relative error of the solution will be magnified by as much as the condition number of the relative error of the right-hand side.

Thus, if the relative error of the right-hand side is 2%, and the condition number is 25, the relative error of what we find to be the solution may be as large as 50%, even if you are using an algorithm that introduces zero numeric error.

Beyond the scope of this course

Suppose that there is also an error in the matrix A, so rather than solving $A\mathbf{u} = \mathbf{v}_0$ for $\mathbf{u}$, we are instead solving

$$(A + \Delta A)\mathbf{u} = \mathbf{v}_0 + \Delta\mathbf{v}.$$

1. Assume that the solution we find is $\mathbf{u}_0 + \Delta\mathbf{u}$.
2. Then $\frac{\|\Delta\mathbf{u}\|_2}{\|\mathbf{u}_0\|_2} \le \frac{\operatorname{cond}(A)}{1-\operatorname{cond}(A)\frac{|||\Delta A|||_2}{|||A|||_2}}\left(\frac{\|\Delta\mathbf{v}\|_2}{\|\mathbf{v}_0\|_2}+\frac{|||\Delta A|||_2}{|||A|||_2}\right)$; that is, the relative error of the solution will be even more magnified by errors in both the right-hand side and the matrix.

In this expression, $|||A|||_2$ is the matrix norm, which is the maximum that the matrix stretches a vector; that is, the maximum possible value of $\frac{\|A\mathbf{u}\|_2}{\|\mathbf{u}\|_2}$. If A is a symmetric matrix, then the norm of the matrix is maximum eigenvalue in absolute value.